2.5 The Rocket Problem

In this section we investigate the thrust that a rocket motor produces by expelling propellant at a high velocity from the spacecraft. Consider the one-stage rocket shown in Figure 2.9. Let m be the mass of the rocket including any propellant that is currently on board. The propellant fuel is being burnt and ejected at a mass flow rate of $\dot{m}$. The current velocity vector of the rocket is $\boldsymbol{v}$, while the exhaust velocity of the ejected propellant particles dm relative to the rocket is $\boldsymbol{v}_e$. Note that the orientation of the exhaust velocity vector $\boldsymbol{v}_e$ does not have to point aftward. If the nozzle would be pointing forward, then the engine would be used to perform a breaking maneuver. The rocket is assumed to be flying through an atmosphere with an ambient pressure P_a. At the point where the exhaust gases escape the engine nozzle the exhaust pressure is given by P_e.

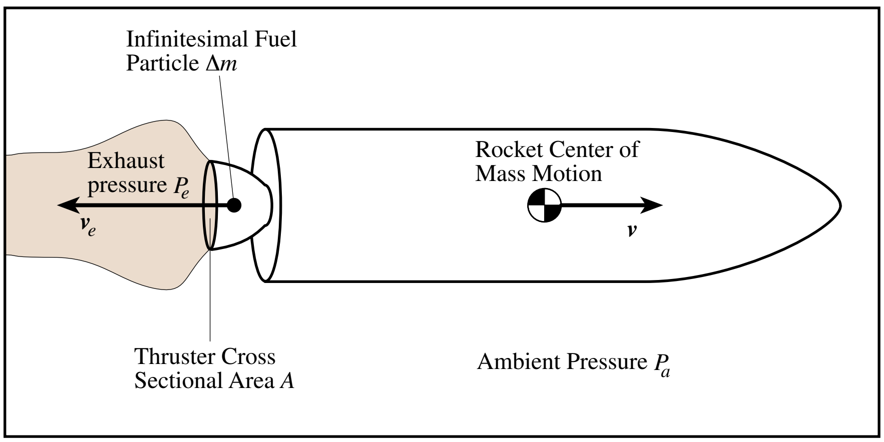

Figure 2.9: A One-Stage Rocket Expelling a Propellant Particle Δm with an Ambient Atmosphere p_a.

We would like to develop the thrust vector that the rocket engine is exerting onto the spacecraft. To do so, we utilize Eq. (2.72) or (2.101) which state that the external force $\boldsymbol{F}$ exerted onto a system of particles or a continuous body is equal time rate of change in linear momentum. Let us treat the rocket mass m and the expelled propellant particle Δm as a two particle system and track their linear momentum change over a small time interval Δt. Using Eq. (2.72) we can write the momentum equation as

$$\boldsymbol{F}\Delta t = \boldsymbol{p}(t + \Delta t) - \boldsymbol{p}(t) \tag{2.102}$$

The quantity $\boldsymbol{F}\Delta t$ is the impulse being applied to the system over the time interval dt. At time t is the rocket and propellant mass is still m. At time $t+\Delta t$, the rocket mass has been reduced to $m - dm$ and the propellant particle Δm is about to leave the engine nozzle. Assume that the only external force acting on this two-particle system is due to pressure differential at the engine nozzle. Let A be the nozzle cross sectional area, then the external force $\boldsymbol{F}$ is expressed as

$$\boldsymbol{F} = -\frac{\boldsymbol{v}_e}{v_e} A\left(P_e - P_a\right) \tag{2.103}$$

More generally, however, we write the external force vector $\boldsymbol{F}$ as

$$\boldsymbol{F} = -\frac{\boldsymbol{v}_e}{v_e} A\left(P_e - P_a\right) + \boldsymbol{F}_e \tag{2.104}$$

where $\boldsymbol{F}_e$ is the net sum of non-pressure related external forces such as gravitational forces acting on the system. The pressure induced force is assumed to be collinear with the exhaust velocity vector $\boldsymbol{v}_e$. Note that if $P_a = P_e$ (exhaust expands to ambient pressure) or $P_a = P_e = 0$ (operating in a vacuum and exhaust expanding to zero pressure), then the net external force on the system is zero. Further, if the direction of the exhaust velocity vector $\boldsymbol{v}_e$ is in the opposite direction to the rocket velocity vector $\boldsymbol{v}$, then a positive pressure differential $P_e - P_a > 0$ results in an acceleration in the rocket velocity direction.

The linear momentum p of the system at time t is

$$\boldsymbol{p}(t) = m\boldsymbol{v} \tag{2.105}$$

since the propellant particle dm is still joined with the rocket. At time $t + \Delta t$ the small propellant mass Δm is being ejected from the rocket with a relative velocity vector $\boldsymbol{v}_e$. Since the rocket is loosing mass, the mass difference Δm over time dt is a negative quantity. The linear momentum at time $t + \Delta t$ is

$$\boldsymbol{p}(t + \Delta t) = (m + \Delta m)(\boldsymbol{v} + \Delta\boldsymbol{v}) - \Delta m(\boldsymbol{v} + \boldsymbol{v}_e) \tag{2.106}$$

where $(m + \Delta m)$ is the rocket mass without the escaping fuel particle and $\Delta\boldsymbol{v}$ is the change in rocket velocity vector over the time interval Δt. Dropping higher order differential terms in Eq. (2.106) and substituting the $\boldsymbol{F}$, $\boldsymbol{p}(t)$ and $\boldsymbol{p}(t+\Delta t)$ expressions into Eq. (2.102) leads to

$$-\frac{\boldsymbol{v}_e}{v_e} A\left(P_e - P_a\right)\Delta t + \boldsymbol{F}_e\Delta t = m\Delta\boldsymbol{v} - \Delta m\boldsymbol{v}_e \tag{2.107}$$

Dividing both sides by Δt and solving for the acceleration term we find

$$m\frac{\Delta\boldsymbol{v}}{\Delta t} = -\frac{\boldsymbol{v}_e}{v_e} A\left(P_e - P_a\right) + \frac{\Delta m}{\Delta t}\boldsymbol{v}_e + \boldsymbol{F}_e \tag{2.108}$$

Allowing the time step Δt to become infinitesimally small, we arrive at the rocket equations of motion:

$$m\frac{d\boldsymbol{v}}{dt} = \underbrace{-\boldsymbol{v}_e\left(\frac{A}{v_e}\left(P_e - P_a\right) - \frac{dm}{dt}\right)}_{\boldsymbol{F}_s} + \boldsymbol{F}_e = \boldsymbol{F}_s + \boldsymbol{F}_e \tag{2.109}$$

The $\boldsymbol{F}_s$ force component is called the *static thrust* of the rocket engine. If the rocket were attached to a test stand, then it would require a force $\boldsymbol{F}_s$ to keep the rocket immobile during the engine test firing.

If the exhaust velocity vector is in the opposite direction to the rocket velocity vector $\boldsymbol{v}$ as shown in Figure 2.9, and the rocket is operating in a weightless environment with $\boldsymbol{F}_e = 0$, then the rocket equations of motion simplify to the famous one-dimensional form

$$m\frac{dv}{dt} = A\left(P_e - P_a\right) - \frac{dm}{dt}v_e = F_s \tag{2.110}$$

with the parameter F_s being the scalar static rocket thrust. Let us assume that over a time interval from t_0 to t_f that the negative mass flow rate $\dot{m}$ is constant. Eq. (2.110) can be rewritten as

$$m\frac{dv}{dm}\dot{m} = F_s \tag{2.111}$$

Rearranging this equation by separating the dv and dm terms, and integrating from t_0 to t_f, we find

$$\int_{v_0}^{v_f} dv = v_f - v_0 = \frac{F_s}{\dot{m}}\int_{m_0}^{m_f}\frac{dm}{m} = -\frac{F_s}{\dot{m}}\ln\left(\frac{m_0}{m_f}\right) \tag{2.112}$$

Note that F_s and $\dot{m}$ can be taken outside the integral sign since they are constants in this investigation. The scalar velocity v_0 is the velocity that the rocket possessed at t_0, while v_f is the rocket velocity at the thruster burnout at t_f. The initial rocket mass is m_0 and the smaller, final rocket mass is m_f. The burn-out velocity v_f can be solve for in terms of the initial rocket velocity and mass, as well as the final burnout mass m_f.

$$v_f = v_0 - \frac{F_s}{\dot{m}}\ln\left(\frac{m_0}{m_f}\right) \tag{2.113}$$

The second term in Eq. (2.113) is a positive quantity since $m_0 > m_f$ and the mass flow rate $\dot{m}$ is a negative quantity. Let $\Delta m < 0$ be the amount of fuel mass lost over the given time interval. Then $m_f = m_0 + \Delta m$. The change in velocity $\Delta v = v_f - v_0$ that results from ejecting Δm of fuel is given by

$$\Delta v = -\frac{F_s}{\dot{m}}\ln\left(\frac{1}{1+\epsilon}\right) \tag{2.114}$$

where $\epsilon = \Delta m/m_0$ is the ratio of fuel spent over the time interval over the initial rocket mass. Note that this change in velocity only depends on the amount of fuel spent and F_s, not on the length of the burning time. Thus, if a thruster produces half the mass flow rate $\dot{m}$ as another thruster, but burns for twice as long, then both thrusters will produce the same velocity change Δv. However, this result is only true if no other external forces are acting on the body. If gravity is pulling on the rocket, then the amount of time spent trying to accelerate the rocket will have a drastic effect on the rocket velocity at burn out time.

A common measure of rocket thruster efficiency is the *specific impulse* I_{sp} defined as[3, 5]

$$I_{sp} = \frac{F_s}{(-\dot{m})g} \tag{2.115}$$

and has units of seconds. The gravitational acceleration g used here is that experienced on the Earth's surface. The higher this I_{sp} value is, the more force the rocket thruster is able to produce for a given mass flow rate. If the exhaust pressure P_e is close to the ambient pressure P_a, the pressure contribution to the static thrust F_s in Eq. (2.110) is negligible. In this case $F_s \approx -\dot{m}v_e$ and the specific impulse simplifies to

$$I_{sp} \approx \frac{v_e}{g} \tag{2.116}$$

From this simplification it is evident that to achieve higher thruster efficiencies, the exhaust velocity v_e should be as high as possible. The faster a given fuel particle is ejected from the rocket, the larger a momentum change (i.e. rocket speed up) it will cause. Using the specific impulse definition, the rocket velocity change Δv for a given fuel ratio ϵ burned is given by

$$\Delta v = I_{sp} g \ln\left(\frac{1}{1+\epsilon}\right) \tag{2.117}$$

The specific impulse ranges for different rocket thruster systems are shown in Table 2.1.[5] Note that the higher specific impulse propulsion methods, such as the ion or arcjet thrusters, typically produce only a very small thrust. Such modes of propulsion are able to achieve a desired Δv with a much smaller amount of fuel mass Δm than a propulsion method with a lower I_{sp}. However, due to the small amount of thrust produced, these efficient propulsion methods will take a much longer time to produce this desired velocity change.

Example 2.9: Assume we are trying to launch an initially at rest sounding rocket vertically from the Earth's surface and it is to only fly several miles high. For these small altitudes, we are still able to assume that the gravitational attraction g is constant during the flight. The solid rocket motor produces a constant I_{sp} for the duration of it burn. Since the only external force acting on the rocket is the constant gravitational acceleration, the rocket equations of motion in the vertical direction are given by Eq. (2.109):

$$m\dot{v} = F_s - mg = g\left(m - I_{sp}\dot{m}\right) \tag{2.118}$$

This equation illustrates the challenge that a highly efficient ion propulsion system would have in attempting to launch this sounding rocket. The change in velocity expression given in Eq. (2.114) assumes that no external forces are acting on the rocket except for the ambient and exhaust pressure. With

Table 2.1: Specific Impulse and Thrust Ranges for Different Rocket Thruster Designs

Thruster Type	Vacuum I_{sp} [sec]	Thrust Range [N]	Comments
Solid Motor	280 – 300	$50 - 5 \cdot 10^6$	Simple, reliable low-cost design with a low performance, but a high thrust
Cold Gas	50 – 75	0.05 – 200	Extremely simple and reliable design with a very low performance and heavy weight for the small thrust produced
Liquid Motor	150 – 450	$5 - 5 \cdot 10^6$	Higher performance thruster at the cost of a more complicated mechanical and cryogenic design
Electrothermal Arcjet	450 – 1500	0.05 – 5	Higher performance, low thrust system with a complicated thermal interface
Ion	2000 – 6000	$5 \cdot 10^{-6} - 0.5$	Very high performance system with typically a very low thrust

the gravity force acting on our sounding rocket, the thruster is constantly battling the gravitational acceleration. In fact, if the rocket thrust is less than the weight mg of the rocket, then the propulsion system will not be able to lift the rocket off the launch pad. Thus, while a ion high performance propulsion system is very effective in accelerating a spacecraft in a weightless or free-falling environment, it would be an inappropriate propulsion choice to launch a rocket of a planet's surface. The rocket velocity at burn out time t_f is then given by

$$v_f = g\left(-t_f + I_{sp}\ln\left(\frac{m_0}{m_f}\right)\right) \tag{2.119}$$

The longer the thruster takes to accelerate the rocket to the desired velocity, the longer the thruster must combat the gravitational acceleration. Since the efficient ion propulsion system requires a large time t_f to achieve a desired Δv, the gravity is also given a large amount of time to counter the achievements of the ion thruster. This is why it is common to use solid or liquid chemical propulsion systems to launch a rocket from the planet's surface to a low-Earth orbit. While these propulsion choices are less efficient, they provide a thrust which is much larger than the rocket weight. With this large static thrust the rocket is propelled to the desired velocity quickly and the gravity field has